International Journal of Theoretical Physics, Vol. 19, No. 12, 1980

Characterization of Quantum Logics

P. J. Lahti

University of Turku, Department of Physical Sciences, SF-20500 Turku 50, Finland

Received May 13, 1980

The quantum logic approach to axiomatic quantum mechanics is used to analyze the conceptual foundations of the traditional quantum theory. The universal quantum of action $h>0$ is incorporated into the theory by introducing the uncertainty principle, the complementarity principle, and the superposition principle into the framework. A characterization of those quantum logics (L, S) which may provide quantum descriptions is then given.

1. INTRODUCTION

Either $h>0$ or $h=0$, exclusively. Physical actions cannot have negative values.

The physical reason for the quantum theoretical description of nature is that Planck's constant h differs from zero.[1] It took about thirty years after Planck's fundamental discovery to successfully integrate the concept of quantum into physics, and it was finally done with the elegant Hilbert space formulation of quantum mechanics due to von Neumann (1932). The consistency of the theory with the existence of the universal quantum of action h was guaranteed by the fact that the formalism was automatically in harmony with the three fundamental quantum principles, namely, with the uncertainty principle, with the complementarity principle, and with the superposition principle. The root of the theory, the h, received here, however, a fairly hidden role, manifesting itself most strikingly in the fundamental "exchange relation" $QP-PQ=ih/2\pi$ but also in the abovementioned three quantum principles.

[1]It has been found empirically that $h=6.6256\times 10^{-34}$ J s.

Since the fundamental work of Mackey (1963) many novel approaches to axiomatic quantum mechanics have been developed—each of them deepening our insight on quantum theory.[2] In particular, the operational approach (Davies and Lewis, 1970; Edwards, 1970) and the convex scheme (Mielnik, 1969, 1974) have proved to be flexible enough to provide us with physically interesting generalizations of the standard quantum mechanics (Mielnik, 1974; Davies, 1976; see also Bugajski, 1979). In spite of the shadows these two approaches have cast on the old quantum logic approach (Mielnik, 1974; Davies, 1976; Bugajski, 1979), this approach appears to provide us with a suitable framework to analyze conceptual foundations of the quantum theory. We thus accept the quantum logic approach here.

It is rather generally agreed that the minimal mathematical structure of any probabilistic physical theory is properly reflected in a couple $(\mathbf{L},\mathbf{S})$, called a quantum logic, where $\mathbf{L}$ carries as a natural structure that of an orthomodular σ-orthocomplete poset and $\mathbf{S}$ is an order-determining set of probability measures on $\mathbf{L}$. In particular, this general setting provides us with the common structure of the classical phase space theory and the quantum Hilbert space theory.

Our concern will be to find out, and to characterize, those quantum logics $(\mathbf{L},\mathbf{S})$ which are of quantum nature. In order to reach such quantum logics from the general frame, which contains both the classical and the quantum case, we have to introduce, either explicitly or implicitly, the universal quantum of action into the theory.

The step from the general $(\mathbf{L},\mathbf{S})$ setting to quantum theory is not of logical but of empirical character. This step consists in introducing the empirical fact of the existence of the universal quantum of action h into the prestructure $(\mathbf{L},\mathbf{S})$, i.e., in quantizing the quantum logic $(\mathbf{L},\mathbf{S})$. We think that *quantization* does not mean, as is usually taught (see, e.g., C. A. Taylor, 1973), introducing discreteness in the spectra of some observables (which results from boundary conditions—either classical or quantal) but rather *introducing the quantum of action h into the theory*. Actually, the quantization in our sense has a consequence of introducing noncompatibility of certain observables, but not the discreteness in their spectra.

Evidently, there are many ways to quantize the $(\mathbf{L},\mathbf{S})$ theory. The first step in this direction was taken by Mackey (1963) with his famous Hilbert space axiom (Mackey's seventh axiom): $\mathbf{L}$ is isomorphic to the projection lattice $\mathbf{P}(\mathbf{H})$ of a complex separable infinite-dimensional Hilbert space $\mathbf{H}$. The second consists of the so-called representation theorems aimed at deriving the above Mackey axiom from a set of "more fundamental"

[2]For a review of the main approaches we refer to Gudder (1977).

assumptions, the most famous of such works being due to Piron (1964). Though there are many novel ways to introduce the Piron structure for **L**, it is to be recognized that in every case (known to us) the fundamental step seems to lack "physical clarity," being only motivated with the acknowledgement of the final goal.[3] Moreover, the crucial branching point between classical theory and quantum theory is usually deeply hidden in the very difficult task of finding physical (operational) motivations for those properties of **L**, like lattice structure or atomicity or covering property or semimodularity, which are essential for the representation theorems, but which do not distinguish the two theories. The introduction of the quantum of action h into the theory is still rather implicit, actually being contained in the claim stressed already by Mackey in his above-cited classic treatise: a quantum theoretic **L** possesses noncompatible propositions, i.e., in the quantum case **L** is non-Boolean. Here we shall propose a more explicit method of quantization.

The quantum principle, the existence of the universal quantum of action h, seems not to provide in itself any quantitative basis for the building up of quantum mechanics from the (**L**, **S**) setting. However, the quantum principle appears itself most strikingly in the inevitable interaction between the object and the instrument during the process of measurement as well as in the fundamental wave–particle duality. Careful investigations of these "quantum anomalies" led Heisenberg (1927), Bohr (1928), and Dirac (1930) to formulate the uncertainty principle, the complementarity principle, and the superposition principle to account for those peculiar features of the experimental situations resulting from the evidence of the quantum. We believe that these three principles comprise the main manifestation of the existence of the universal quantum of action h.

Though we leave it open whether the above-mentioned three principles jointly exhaust the quantum principle we propose that *the quantization of the* (**L**, **S**) *theory consists of formulating the uncertainty principle, the complementarity principle, and the superposition principle in that general frame and picking out those quantum logics* (**L**, **S**) *which satisfy the principles.*

The suggested method of quantization leads also to a natural classification of quantum logics, which, we think, adds to our understanding of the conceptual foundations of the quantum theory.

In order to reach a more satisfactory characterization of the quantum logics providing quantum descriptions one should also consider the formalization of measurements and the questions of dynamics within the

[3] By the Piron structure of **L** we mean the structure of a complete atomic orthocomplemented orthomodular lattice with covering property. For a detailed account of the different approaches to produce this structure for **L** we refer to Beltrametti and Cassinelli (1976, 1979).

framework. However, these questions lie outside the scope of the present paper.[4]

2. A QUANTUM LOGIC—THE GENERAL FRAME

In order to fix our general framework we accept the quantum logic approach to nonrelativistic axiomatic quantum theory. In this approach it is assumed that the minimal mathematical structure of any nonrelativistic probabilistic (irreducibly or otherwise) physical theory is properly reflected in a couple $(\mathbf{L},\mathbf{S})$, where $\mathbf{L}$, the set of all *propositions* on the physical system concerned, carries as a natural structure that of an orthomodular σ-orthocomplete partially ordered set and $\mathbf{S}$ is an order-determining or full set of probability measures, *states*, on $\mathbf{L}$. Such a prestructure or skeleton $(\mathbf{L},\mathbf{S})$ is called a *quantum logic*. The family of all quantum logics is denoted by $\mathfrak{L}$.

In this approach the *observables* of the considered physical system are described as $\mathbf{L}$-valued measures on $(R,\mathbf{B}(R))$, where $\mathbf{B}(R)$ denotes the set of all Borel subsets of the real line R. The probability measure

$$\alpha\circ A:\mathbf{B}(R)\to[0,1],\qquad E\to\alpha(A(E))$$

is interpreted as the probability distribution of the observable A in the state α. Moreover, the set of all observables $\mathbf{O}$ of the physical system concerned is *surjective*: for each a in $\mathbf{L}$ there is an A in $\mathbf{O}$ and an E in $\mathbf{B}(R)$ such that $a=A(E)$.

For a general exposition of this approach, as well as for the standard definitions of the concepts like atom, pure state, $\mathrm{sp}(A)$ (the spectrum of the observable A), $\mathrm{Var}(A,\alpha)$ (the variance of A in the state α) appearing in the text, the reader is referred to Mackey (1963), Maczynski (1967, 1973), Varadarajan (1968), Gudder (1970), and Beltrametti and Cassinelli (1976).

3. $\mathfrak{L}_{cl}$—THE CLASSICAL CASE

The classical Hamiltonian description of a physical system with n *degrees of freedom* is carried out in the *phase space* M ($2n$-dimensional real Euclidean space) whose Borel structure $\mathbf{B}(M)$ describes in a natural way the set of all propositions on the system. We denote this set as $\mathbf{L}$. *The*

[4]A formalization of the ideal, first-kind measurements has already been successfully done within the quantum logic approach (Pool, 1968; Beltrametti and Cassinelli, 1976, 1977; Cassinelli and Beltrametti, 1975). The question of dynamics is not of great relevance here because we are dealing with nonrelativistic theory, in which the questions of dynamics and statistics can be considered independently.

dynamic variables of the system are expressed as real-valued Borel functions on **M**. The σ homomorphisms induced by the dynamic variables are called the observables of the physical system, the set of all observables $\{A:\mathbf{B}(R)\to\mathbf{L},\ A\ \sigma \text{ homomorphism}\}$ being denoted as **O**. The points of **M** describe the states of the system. The points of **M** can be identified with the unit measures on **L**. To allow also nontrivial probability measures on **L** as states of the system (mixed states) we generalize thus: a state of the physical system is a probability measure on **L**. We denote the set of all such states $\{\alpha:\mathbf{L}\to[0,1],\ \alpha \text{ probability measure}\}$ as **S**. The probability measure $\alpha\circ A:\mathbf{B}(R)\to\mathbf{L}$ gives the probability distribution of the observable A in the state α.

The classical Hamiltonian description of a physical system with n degrees of freedom is thus based on a quantum logic $(\mathbf{L},\mathbf{S})$ with **L** as the Borel structure of the $2n$-dimensional phase space **M** and **S** as the set of all probability measures on **L**. We define

$$\mathfrak{L}_{\mathrm{cl}}=\begin{array}{l}\{(\mathbf{L},\mathbf{S})\in\mathfrak{L}\,|\,\mathbf{L}=\mathbf{B}(\mathbf{M}),\ \mathbf{M} \text{ is a phase space, } \mathbf{S} \text{ is the set of}\\ \text{all probability measures on } \mathbf{B}(M)\}\end{array} \tag{1}$$

In the classical case, i.e., when $(\mathbf{L},\mathbf{S})\in\mathfrak{L}_{\mathrm{cl}}$, **L** is Boolean, and the set **P** of all pure states (unit measures) on **L** is *sufficient*, satisfying thus *the principle of plenitude*: for each a in **L**, $a\neq 0$, there is a pure state p in **P** such that $p(a)=1$. Moreover, any α in **S** can uniquely be expressed as a (countable or otherwise) mixture of pure states in **S**.

4. $\mathfrak{L}_{\mathrm{SP}}$—THE SUPERPOSITION PRINCIPLE SP

In the quantum logic approach two natural generalizations of Dirac's notion of superposition, and thus of the superposition principle, arise:[5] the one referring to the *atoms* of **L**, the other referring to the *pure states* of **S**. Thus in order to formulate this notion and to state the principle in the $(\mathbf{L},\mathbf{S})$ setting we have to assume that **L** contains atoms or **S** contains pure states.

[5] In *Dirac*'s approach the mathematical formulation of the superposition principle led to the requirement of the linearity of the state space (Dirac, 1930). For a recent account of some experiments supporting the superposition principle we refer to Gerjuoy (1973). Of course the validity of the superposition principle as a law of nature is restricted both with the discovery of superselection rules (Wick et al., 1952) and with the evidence calling for nonlinear generalizations of quantum mechanics (see, e.g., Mielnik, 1974; Haag and Bannier, 1978; Bugajski, 1979).

The following two independent notions of superposition are frequently discussed:[6]

A pure state α is a *superposition of pure states* α_1 and α_2 iff $\alpha_1(a)=\alpha_2(a)=0$ implies $\alpha(a)=0$ for every a in **L**.

An atom a is a *superposition of atoms* a_1 and a_2 iff $a \leqslant b$ for every b in **L** such that $b \geqslant a_1$ and $b \geqslant a_2$.

The resulting two forms of the superposition principle can be put forward as follows:

Definition 1. The quantum logic $(\mathbf{L},\mathbf{S})$ satisfies *the state-theoretic superposition principle* if for any two distinct pure states α_1 and α_2 there is a third pure state α_3, distinct from α_1 and α_2, which is their superposition.

Definition 2. The quantum logic $(\mathbf{L},\mathbf{S})$ satisfies *the lattice-theoretic superposition principle* if for any two distinct atoms a_1 and a_2 there is a third atom a_3, distinct from a_1 and a_2, which is their superposition.

We proceed by defining

$$\mathfrak{L}_{SP} = \begin{array}{l}\{(\mathbf{L},\mathbf{S}) \in \mathfrak{L} \mid (\mathbf{L},\mathbf{S}) \text{ satisfies either the lattice-theoretic or} \\ \text{the state-theoretic superposition principle}\}\end{array} \qquad (2)$$

We do not lack quantum logics which satisfy the superposition principle. Moreover, it is most evident that any $(\mathbf{L},\mathbf{S})$ in $\mathfrak{L}_{cl}$ can satisfy neither the lattice- nor the state-theoretic formulation of this principle. Thus we have the following theorem.

Theorem 1. $\mathfrak{L}_{SP} \neq \varnothing$. $\mathfrak{L}_{cl} \cap \mathfrak{L}_{SP} = \varnothing$.

According to Jauch (1968) the essential feature of quantum mechanics is the existence of noncompatible propositions, which means that the structure of a quantum proposition system is non-Boolean. Giving a lattice-theoretic formulation for the superposition principle Jauch shows that a lattice satisfying this principle cannot be Boolean, hence the superposition principle implies the essential feature of quantum mechanics. According to Jauch this shows that the superposition principle is not a new axiom of quantum mechanics but it is merely a consequence of the non-Boolean structure of the proposition system. We, however, advocate the converse view. The non-Boolean structure of **L** is not the most important manifestation of the quantum nature of the system considered.

[6]For details and for original references see Bugajski and Lahti (1980).

The non-Boolean structure of **L** is an important consequence of, e.g., the lattice-theoretic formulation of the superposition principle. But, as we shall see, there are quantizations of the prestructure (**L**, **S**) which do not lead to non-Boolean **L**. Moreover, we have abundance of (**L**, **S**) systems with non-Boolean **L** which seem to lack any relevance to the present physics. [See Mielnik (1968) for similar views.]

5. $\mathfrak{L}_{UP}$—THE UNCERTAINTY PRINCIPLE UP

In the Hilbert space quantum theory the uncertainty principle, the first of the two principles leading to the Copenhagen clarification of the conceptual foundations of the quantum theory, appears itself most strikingly in the uncertainty relation for the canonical position and momentum observables Q and P:

$$\mathrm{Var}(Q,\phi)\cdot\mathrm{Var}(P,\phi)\geqslant(h/4\pi)^2 \qquad (\phi\in\mathrm{dom}(QP)\cap\mathrm{dom}(PQ))$$

The uncertainty principle is thought to provide the consistency of the essentially probabilistic state description of quantum mechanics. It is a characteristic feature of quantum mechanics that only such states of a physical system can be prepared for which the product of the "uncertainties" (i.e., standard deviatiations) of any pair of conjugate variables Q and P has a lower bound given by $h/4\pi$. This is what *Heisenberg* advocated as a direct intuitive interpretation of the fundamental "exchange relation."[7]

In the quantum logic approach we lack any analogy of the Schwarz inequality which led to the above uncertainty relation. However, it appears to us that the very idea of the uncertainty principle expressed above can easily be translated to the quantum logic setting, too. The following form for this principle in quatum logic seems to be acceptable (Lahti, 1979a, 1980).

Definition 3. The quantum logic (**L**, **S**) satisfies the uncertainty principle if there exist at least two observables A and B in **O** and a positive number h, such that for every state α in **S**, for which the variances of A and B are well defined, the inequality

$$\mathrm{Var}(A,\alpha)\cdot\mathrm{Var}(B,\alpha)\geqslant h$$

holds.

We define

$$\mathfrak{L}_{UP}=\{(\mathbf{L},\mathbf{S})\in\mathfrak{L}\,|\,(\mathbf{L},\mathbf{S})\text{ satisfies the uncertainty principle}\} \qquad (3)$$

[7]For a historicocritical analysis of this subject matter refer to Jammer (1974).

Again, we do not lack quantum logics satisfying the uncertainty principle. Moreover, it is clear that an $(\mathbf{L},\mathbf{S})$ in $\mathcal{L}_{\text{cl}}$ cannot satisfy the uncertainty principle. Thus we have the following theorem.

Theorem 2. $\mathcal{L}_{\text{UP}} \neq \varnothing$. $\mathcal{L}_{\text{cl}} \cap \mathcal{L}_{\text{UP}} = \varnothing$.

In addition to the above result it is known that whenever the state system $\mathbf{S}$ satisfies the principle of plenitude the observables A and B satisfying the UP are unbounded and noncompatible (Lahti, 1979a, 1980). Thus for any $(\mathbf{L},\mathbf{S})$ in $\mathcal{L}_{\text{UP}}$, with sufficient $\mathbf{S}, \mathbf{L}$ is non-Boolean.

6. $\mathcal{L}_{\text{CP}}$—THE COMPLEMENTARITY PRINCIPLE CP

The second of the two principles which led to the Copenhagen solution of the interpretation problem of the quantum theory is the complementarity principle, by *Bohr*. The standard Hilbert space quantum theory is built up in such a way that it is automatically in harmony with the superposition principle and with the uncertainty principle, but also with the complementarity principle.[8] Really, the Bohrian view that position and momentum Q and P are complementary observables is properly reflected in the Hilbert space result

$$P^Q(E) \wedge P^P(F) = 0 \qquad \text{for any bounded } E \text{ and } F \text{ in } \mathbf{B}(R)$$

which results from the Fourier–Plancherel equivalence of Q and P. Here P^Q and P^P denote the spectral measures of Q and P, respectively.

We proceed by abstracting the above Hilbert space expression of complementarity of certain observables to the quantum logic frame.

Definition 4. Observables A and B are *complementary* if for any bounded Borel sets E and F such that $E \cap \text{sp}(A) \subsetneq \text{sp}(A)$ and $F \cap \text{sp}(B) \subsetneq \text{sp}(B)$, the lattice meet $A(E) \wedge B(F)$ exists in $\mathbf{L}$ and equals the least element of $\mathbf{L}$, 0.

Definition 5. The quantum logic $(\mathbf{L},\mathbf{S})$ satisfies *the complementarity principle* if there exist in $\mathbf{O}$ at least two nonconstant complementary observables.

[8] Bohr's viewpoint of complementarity is not very compact, and it does not contain any clear definition. However, the most obvious fact is that "complementarity is a binary relationship: some A is complementary to some B" (Scheibe, 1973; see also Jammer, 1974, and Lahti, 1979a). We follow Bohr's important paper (Bohr, 1935) formulating complementarity as a binary relationship on the set of all observables $\mathbf{O}$. Intuitively, we say that observables A and B are complementary if the experimental arrangements which permit their unambiguous definitions are mutually exclusive.

We define

$$\mathfrak{L}_{CP} = \{(\mathbf{L},\mathbf{S}) \in \mathfrak{L} \mid (\mathbf{L},\mathbf{S}) \text{ satisfies the complementarity principle}\} \quad (4)$$

Again, we do not lack quantum logics satisfying the complementarity principle. Moreover, it is evident that an $(\mathbf{L},\mathbf{S})$ in $\mathfrak{L}_{cl}$ cannot satisfy the complementarity principle. Thus we have the following theorem.

Theorem 3. $\mathfrak{L}_{CP} \neq \varnothing$. $\mathfrak{L}_{cl} \cap \mathfrak{L}_{CP} = \varnothing$.

In fact, in (Lahti, 1979a, 1980) it is shown that nonconstant complementary observables are noncompatible, which means that for any $(\mathbf{L},\mathbf{S})$ in $\mathfrak{L}_{CP}$ $\mathbf{L}$ is non-Boolean.

In his papers (1963, 1969) Finkelstein also stressed the foundational status of complementarity in quantum theory, and gave a formulation for this notion in the quantum logic scheme. According to Finkelstein (1963), two propositions (actually atoms) a and b are complementary if

$$a \neq (a \wedge b) \vee (a \wedge b^{\perp}) \quad (5)$$

One immediately realizes that our CP implies the existence of complementary propositions in Finkelstein's sense, but that the two notions are not equivalent. Really, our CP implies the existence of such propositions a and b in $\mathbf{L}$ for which $a \wedge b = 0$ but $a \not\leq b^{\perp}$. On the other hand, $L(R^3)$ (see Section 9.1) possesses complementary propositions in Finkelstein's sense, but it does not admit any complementary observables.

As (5) leads to a nondistributive $\mathbf{L}$, Finkelstein emphasized complementarity as an instance of nondistributivity. More fundamental in Finkelstein's approach is coherence: $\mathbf{L}$ is coherent if for any pair of disjoint atoms a and b in $\mathbf{L}$, there is a third atom c in $\mathbf{L}$, disjoint from a and b, such that

$$a \vee b = b \vee c = c \vee a \quad (6)$$

Such a c he calls a coherent superposition of a and b, a notion which agrees with Jauch's notion of superposition (see Jauch 1968). Thus in Finkelstein's scheme coherence (i.e., superposition) implies complementarity. Finkelstein ended with his formulations in discussing the "photon polarization" or the "spin-1/2 particle in a Stern–Gerlach apparatus" lattices (see Section 9.3). These are examples in which the notions of coherence and complementarity are essentially the same. We note that in those examples also our notions of superposition and complementarity coincide, but we think that this coincidence is only accidental— being due to the specific nature of the examples considered.

7. DIGRESSION—COMPLEMENTARITY BREAKS MODULARITY?

According to Jauch (1968) and Finkelstein (1963, 1969), coherence is the essential quantum feature. Coherence of **L** implies the nondistributivity of **L**. The superposition principle and complementarity, which are characteristic features of quantum mechanics, are said to be special instances of the nondistributivity of **L**. Moreover, if the length of the unit element of **L**, say n, is greater than 2, then a coherent **L** is essentially isomorphic to the lattice $\mathbf{L}(V(n))$ of all vector subspaces of an n-dimensional vector space $V(n)$. [See Finkelstein (1963) and Jauch (1968).] This is essentially the content of the celebrated Piron representation theorem (Piron, 1964).) So let us consider briefly the vector space models of **L**.

If $\mathbf{L} \cong \mathbf{L}(V(1))$, then L is distributive and does not have (nontrivial) coherence. If $\mathbf{L} \cong \mathbf{L}(V(n))$, $n \geqslant 2$, then **L** has coherence, and is thus nondistributive, actually modular whenever $n < \infty$.

In the case $\mathbf{L} \cong \mathbf{L}(V(2))$, both SP and CP hold. Actually, in this case the notions of superposition and complementarity coincide. In the case $\mathbf{L} \cong \mathbf{L}(V(3))$ we still have superpositions, but we do not have any complementary observables. In the case $\mathbf{L} \cong \mathbf{L}(V(n))$, with $n \geqslant 4$, SP and CP hold again, but the two notions do not coincide any more.

We call an observable $A : \mathbf{B}(R) \to \mathbf{L}$ *maximal* if whenever $a \in A(\mathbf{B}(R))$ and $p \leqslant a$ is an atom, then also $p \in A(\mathbf{B}(R))$. One immediately recognizes that the existence of maximal complementary observables is inconsistent with the finite-dimensional ($n \geqslant 3$) vector space models for **L**. We note that position and momentum are the most important examples of complementary observables. They are maximal. Thus it appears to us that complementarity is responsible for the relaxation of the modularity of **L**.

To conclude the above findings, we note that the (**L**, **S**) theory supplemented with coherence is quantal, leading to a vector space model for **L**. When the (**L**, **S**) theory is supplemented both with coherence and with maximal complementarity, we end with infinite-dimensional vector space models for **L**. Thus coherence leads to the break of distributivity, and maximal complementarity leads to the break of modularity.

8. $\mathfrak{L}_{\mathrm{DHB}}$ AND $\mathfrak{L}_{\mathrm{vN}}$

The three fundamental principles of the quantum theory are, as agreed, the superposition principle (due to *Dirac*), the uncertainty principle (due to *Heisenberg*), and the complementarity principle (due to *Bohr*). In the preceding sections it was shown that the natural formulations of these principles in the quantum logic setting are of quantum nature, each of them excluding the classical description. In the next chapter we shall discuss some familiar quantum logics which, among other things, indicate the mutual independence of the principles.[9]

Owing to the foundational status of these principles in the quantum theory we suggest the following: *A necessary condition for* a given quantum logic $(\mathbf{L},\mathbf{S})$ to provide *a quantum description* of any physical system is that $(\mathbf{L},\mathbf{S})$ satisfies either the superposition principle or the uncertainty principle or the complementarity principle. In other words, quantum logics $(\mathbf{L},\mathbf{S})$ in

$$\mathfrak{L}_{\mathrm{qu}}=\mathfrak{L}_{\mathrm{SP}\vee\mathrm{UP}\vee\mathrm{CP}}=\mathfrak{L}_{\mathrm{SP}}\cup\mathfrak{L}_{\mathrm{UP}}\cup\mathfrak{L}_{\mathrm{CP}} \tag{7}$$

and only those, can provide quantum descriptions.

Among the most interesting subclasses of $\mathfrak{L}_{\mathrm{qu}}$ are the following two. The Dirac–Heisenberg–Bohr quantum logics, i.e., those quantum logics which satisfy all the three quantum principles:

$$\mathfrak{L}_{\mathrm{DHB}}=\mathfrak{L}_{\mathrm{SP}\wedge\mathrm{UP}\wedge\mathrm{CP}}=\mathfrak{L}_{\mathrm{SP}}\cap\mathfrak{L}_{\mathrm{UP}}\cap\mathfrak{L}_{\mathrm{CP}} \tag{8}$$

and the von Neumann quantum logics or the Hilbertian quantum logics:

$$\mathfrak{L}_{\mathrm{vN}}=\{(\mathbf{L},\mathbf{S})\in\mathfrak{L}\,|\,(\mathbf{L},\mathbf{S})\text{ is a Hilbertian quantum logic}\}. \tag{9}$$

By a Hilbertian quantum logic $(\mathbf{L},\mathbf{S})$ we mean a quantum logic which gives us the standard Hilbert space quantum theory.

The relevance of the distinction between (8) and (9) lies in the fact that (see Bugajski and Lahti, 1980) any Hilbertian quantum logic is a Dirac–Heisenberg–Bohr quantum logic, but not conversely.

9. SOME QUANTUM LOGICS—EXAMPLES

Next we shall study some frequently appearing quantum logics which on the one hand confirm some of the above statements and on the other hand give rise to further considerations.

9.1. The Quantum Logic $(\mathbf{L}(R^3),\mathbf{S})$. Consider the quantum logic $(\mathbf{L}(R^3),\mathbf{S})$ with $\mathbf{L}(R^3)$ as the lattice of all subspaces of the three-dimensional real Euclidean space R^3, and with $\mathbf{S}$ as the set of all Gleason states on $\mathbf{L}(R^3)$. $\mathbf{L}(R^3)$ is a lattice containing atoms, atoms being one-dimensional subspaces of R^3. According to Gleason (1957) each state on $\mathbf{L}(R^3)$ is either

[9]This matter is discussed in more detail in (Lahti, 1979b).

a pure state (induced by a unit vector of R^3) or a mixture of pure states in **S**. Moreover, the pure states on $\mathbf{L}(R^3)$ are in natural one-to-one correspondence with the atoms of $\mathbf{L}(R^3)$.

SP. One immediately recognizes that $(\mathbf{L}(R^3), \mathbf{S})$ satisfies both the lattice-theoretic and the state-theoretic superposition principle. Moreover, in this case the two notions of superposition are essentially the same.

CP. Let A and B be two nonconstant observables, i.e., σ homomorphisms $\mathbf{B}(R) \to \mathbf{L}(R^3)$ with nontrivial ranges. This means that both the ranges $A(\mathbf{B}(R))$ and $B(\mathbf{B}(R))$ contain two-dimensional subspaces of R^3. The intersection of any two-dimensional subspaces of R^3 is, at least, one-dimensional subspace. Thus A and B cannot be complementary, i.e., $(\mathbf{L}(R^3), \mathbf{S})$ does not satisfy the complementarity principle.

UP. **S** is sufficient, now. Thus the observables A and B satisfying the inequality

(a) $$\mathrm{Var}(A, \alpha) \cdot \mathrm{Var}(B, \alpha) \geqslant h \qquad \forall \alpha \in S_A^V \cap S_B^V$$

are unbounded (i.e., their spectra are unbounded) and noncompatible.[10] But they cannot be, as indicated above, complementary. This means that there exist two bounded Borel sets E and F of the real line R such that $0 < A(E) \wedge B(F) < 1$, $A(E) \neq 1$, $B(F) \neq 1$. Let α in **S** be such that $\alpha(A(E) \wedge B(F)) = 1$. In this state α we have

(b) $$\mathrm{Var}(A, \alpha) \cdot \mathrm{Var}(B, \alpha) < \nu(E) \cdot \nu(F)$$

with the notation $\nu(E) = \sup\{x^2 : x \in E\} - \inf\{|x| : x \in E\}^2$. The claims (a) and (b) are to hold simultaneously, which means a limitation to the "sizes" of the above like bounded sets E and F:

(ab) $$\nu(E) \cdot \nu(F) \geqslant h$$

To conclude: The quantum logic $(\mathbf{L}(R^3), \mathbf{S})$ may provide a quantum description of some physical system—a description in which the superposition principle holds, but the complementarity principle does not hold. An actual application of the structure $(\mathbf{L}(R^3), \mathbf{S})$ could be provided, e.g., with a description of a spin-1 particle.

9.2. The Quantum Logic $(D_{16}, \bar{P})$. We consider now the well-known 16-element orthomodular lattice D_{16},[11] with a natural state system $\bar{P}$. The Hasse diagram of D_{16} is given in Figure 1.

[10] S_A^V denotes the set of those states α in **S** for which $\mathrm{Var}(A, \alpha)$ exists and is finite.

[11] See, e.g., Greechie and Gudder (1973).

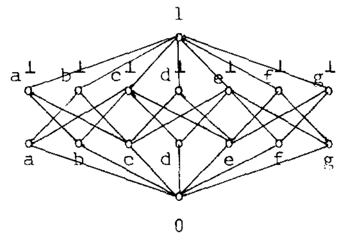

Fig. 1. The Hasse diagram of the orthomodular lattice D_{16}.

SP. b and c are two distinct atoms of D_{16}, but there is no atom in D_{16} which would be their superposition. Thus the lattice-theoretic superposition principle does not hold in D_{16}.

D_{16} admits a great amount of full sets of states on it. For example, D_{16} admits a full and sufficient set P of (pure) 0-1 states on it such that the natural one-to-one correspondence between the atoms of D_{16} and the pure states of P holds: fix α_a such that carr$\alpha_a = \{x \in D_{16} : \alpha_a(x) = 1\} = a^{\nabla} = \{x \in D_{16} : a \leqslant x\}$, and put $\alpha_a(x) = 0$ otherwise; similarly for the remaining six atoms of D_{16} (Bugajska and Bugajski, 1973). No pure state in P is a superposition of some other pure states in P. This "classical" P is, however, easily modified to a $\bar{P}$ such that the fullness and the sufficiency as well as the above one-to-one correspondence between the elements of $\bar{P}$ and the atoms of D_{16} is maintained, and that exactly those (corresponding) superpositions hold in $\bar{P}$ which hold in D_{16}. Evidently, one cannot construct a full set of states **S** on D_{16} such that the atoms of D_{16} correspond one-to-one with the pure states of **S**, and that the superposition principle holds in **S**.

CP. The observables A and G defined through the ranges $\{0, a, a^{\perp}, 1\}$ and $\{0, g, g^{\perp}, 1\}$ with spectra $\{\lambda_1, \lambda_2\}$, $\{\mu_1, \mu_2\}$ are complementary. Thus the quantum logic $(D_{16}, \mathbf{S})$, with any full **S**, satisfies the complementarity principle.

UP. It is most evident that the quantum logic $(D_{16}, \mathbf{S})$, with any sufficient **S**, does not satisfy the uncertainty principle

To conclude: $(D_{16}, \bar{P})$ may provide a quantum description of some physical system—a description in which the complementarity principle holds but the other two principles do not. However, we do not know any actual application of the quantum logic in question.

9.3. The "Photon Polarization" Quantum Logic (L, P). We discuss next the "photon polarization" quantum logic (**L**, **P**) with **L** and **P** defined in the natural way as $\mathbf{L} = \{0, 1, a_{\phi}, a_{\phi}^{\perp} = a_{\phi+\pi/2} : 0 \leqslant \phi < \pi/2\}$ and $\mathbf{P} = \{\alpha_{\phi} : \alpha_{\phi}(a_{\Psi}) = \cos(\phi - \Psi)^2\}$. **L** is a nondistributive lattice containing atoms;

in fact each nontrivial element of **L** is an atom. The set **P** of pure states on **L** is full and sufficient, and the natural one-to-one correspondence between atoms and pure states holds.

SP. The quantum logic (**L**,**P**) satisfies both the lattice-theoretic and the state-theoretic formulations of the superposition principle.

CP. Any two distinct observables with ranges $\{0, a_\phi, a_\phi^\perp, 1\}$ and $\{0, a_\Psi, a_\Psi^\perp, 1\}$ are complementary, so that (**L**,**P**) satisfies the complementarity principle.

UP. In the present case any observable in **O** is of the form $A_\phi = \{0, a_\phi, a_\phi^\perp, 1\}$, $\mathrm{sp}(A_\phi) = \{\lambda_{a_\phi}, \lambda_{a_\phi^\perp}\}$. Thus in any state α_Ψ in **P** $\mathrm{Var}(A_\phi, \alpha_\Psi) = (\lambda_{a_\phi} - \lambda_{a_\phi^\perp})^2 \cos(\Psi - \phi)^2 \cdot \sin(\Psi - \phi)^2$. Though $\mathrm{Var}(A_\phi, \alpha_\Psi) \cdot \mathrm{Var}(A_{\phi'}, \alpha_\Psi) > 0$ for any α_Ψ in **P** such that $\phi \neq \Psi \neq \phi'$, the uncertainty principle does not hold in (**L**,**P**).

To conclude: The quantum logic (**L**,**P**) satisfies the superposition principle and the complementarity principle, but not the uncertainty principle. Thus, it may provide a quantum description of some physical system. In fact, the present quantum logic describes, e.g., the photon polarization experiments with different polarization angles ϕ and $\Psi (\phi \neq \Psi \bmod \pi)$. Another application of this type of quantum logic would be a description of a spin-1/2 particle in the Stern–Gerlach apparatus.

9.4. The Quantum Logic (B(M), S$_h$). In Section 3 we recalled the classical Hamiltonian description of a physical system with n degrees of freedom. For convenience, let $n=1$ now, so that the classical description of the system concerned is based on the quantum logic (**L**, **S**) with $\mathbf{L} = \mathbf{B}(R^2)$ = the Borel family of the two-dimensional phase space $\mathbf{M} = R^2$ and with **S** as the set of all probability measures on $\mathbf{B}(R^2)$. The conjugate position and momentum observables Q and P are now induced by the (conjugate) position and momentum coordinates f_q and f_p [i.e., $Q(E) = f_q^{-1}(E)$ for every E in $\mathbf{B}(R)$, with $f_q: R^2 \to R, (q, p) \to f_q(q, p) = q$; similarly for f_p and P].

The set **S** describes, actually, the set of all physically possible state preparations. Let us assume that only such state preparations are now physically possible which fulfill the requirement of the uncertainty principle. So we define

$$\mathbf{S}_h = \{\alpha \in S: \mathrm{Var}(Q, \alpha) \cdot \mathrm{Var}(P, \alpha) \geqslant (h/4\pi)^2\}$$

The set $\mathbf{S}_h$ is a full set of probability measures on $\mathbf{B}(R^2)$.[12] Of course, $\mathbf{S}_h$ does not satisfy the principle of plenitude, i.e., it is not sufficient.

[12] This is demonstrated by Bugajski (private communication).

SP. It is clear that the quantum logic $(\mathbf{B}(R^2), \mathbf{S}_h)$ satisfies neither the lattice-theoretic nor the state-theoretic formulation of the superposition principle.

CP. For any two bounded Borel sets E and F in $\mathbf{B}(R)$ $Q(E) \wedge P(F) = E \times F$. Thus $(\mathbf{B}(R^2), \mathbf{S}_h)$ does not satisfy the complementarity principle.

UP. The canonically conjugate position and momentum observables Q and P satisfy, by the very definition of $\mathbf{S}_h$, the uncertainty relation. Thus $(\mathbf{B}(R^2), \mathbf{S}_h)$ satisfies the uncertainty principle.

To conclude. The quantum logic $(\mathbf{B}(\mathbf{M}), \mathbf{S}_h)$ may provide a quantum description of some physical system—a description in which the uncertainty principle holds, but the other two principles do not. Though it is obvious that this kind of description does not provide us, e.g., with a correct picture of a hydrogen atom, there is a lesson to be learned from the present quantum logic. If the uncertainty principle would be enough to take care of the physical fact that $h > 0$, then the classical phase space description implemented with the requirement $h > 0$, i.e., restricting $\mathbf{S}$ to $\mathbf{S}_h$, should work as the quantum theory. But it does not. The fact that $(\mathbf{B}(R^2), \mathbf{S}_h)$ is a quantum description with Boolean proposition system $(\mathbf{B}(R^2)$ shows that the common claim that "the transition from classical to quantum mechanics is to be understood as the transition from a Boolean to a non-Boolean possibility structure of events" (Bub, 1979) is not to be taken literally. Finally, we note that in the present case *the correspondence principle* receives a natural, and explicit, formulation in the fact that $\lim_{h \to 0} \mathbf{S}_h = \mathbf{S}$ (i.e., $\mathbf{S}_h \subseteq \mathbf{S}_{h'}$ whenever $h' \leqslant h$). There seems to be no way of doing the same with the proposition systems, i.e., the transformation non-Boolean $\mathbf{L}_h \to$Boolean $\mathbf{L}$, as $h \to 0$, is not so obvious. In fact, it is unknown to us.

9.5. The Quantum Logic $(J_{18}, \mathbf{P})$. Our last example will be the quantum logic $(J_{18}, \mathbf{P})$ with J_{18} as the 18-element orthoposet and with $\mathbf{P}$ as a full and sufficient set of pure states on J_{18}. J_{18} is the first known orthomodularposet which is not a lattice (Janowitz, 1963; see also Greechie, 1969). Again, one can choose $\mathbf{P}$ such that the natural one-to-one correspondence between the atoms of J_{18} and the elements of $\mathbf{P}$ holds, and that exactly those superpositions hold in $\mathbf{P}$ which hold in J_{18}.

SP. $(J_{18}, \mathbf{P})$ satisfies neither the lattice-theoretic nor the state-theoretic formulations of the superposition principle.

CP. $(J_{18}, \mathbf{P})$ does not satisfy the complementarity principle.

UP. $(J_{18}, \mathbf{P})$ does not satisfy the uncertainty principle.

To conclude. The quantum logic $(J_{18}, \mathbf{P})$ does not provide a quantum description of any physical system—none of the three quantum principles holds in $(J_{18}, \mathbf{P})$. Moreover, it is evident that this quantum logic cannot provide a classical description of any physical system. Thus $(J_{18}, \mathbf{P})$, being in $\mathfrak{L}\backslash(\mathfrak{L}_{cl} \cup \mathfrak{L}_{qu})$, seems to have no relevance to physics. Thus, e.g., the lattice assumption of **L** cannot be questioned by referring to J_{18} which is not a lattice. We now also appreciate that the common claim "**L** is not distributive and therefore corresponds to a quantum system" (Greechie and Gudder, 1973) cannot be taken literally.

9.6. Some Remarks on $\mathfrak{L}_{DHB}$ and $\mathfrak{L}_{vN}$. In Section 7 we made the distinction between the Dirac–Heisenberg–Bohr quantum logics and the von Neumann quantum logics, referring to the fact that any Hilbertian quantum logic is a Dirac–Heisenberg–Bohr quantum logic, but not conversely.

The Hilbert space quantum theory is built up in such a way that it is automatically in harmony with the three important quantum principles. This means that $\mathfrak{L}_{vN} \subseteq \mathfrak{L}_{DHB}$. In Bugajski and Lahti (1980) it is demonstrated that the horizontal sum of any two Hilbertian quantum logics is a DHB quantum logic but not a vN quantum logic, establishing the above-stated fact.

10. DISCUSSION

In the preceding sections we defined the following subsystems of the set $\mathfrak{L}$ of all quantum logics $(\mathbf{L}, \mathbf{S})$: $\mathfrak{L}_{cl}$, $\mathfrak{L}_{SP}$, $\mathfrak{L}_{UP}$, $\mathfrak{L}_{CP}$, $\mathfrak{L}_{qu}$, $\mathfrak{L}_{DHB}$, and $\mathfrak{L}_{vN}$. We shall now summarize the most important properties of these sets and discuss their mutual connections. We denote by $\mathfrak{L}_\alpha^\perp$ the set theoretic complement of $\mathfrak{L}_\alpha$ in $\mathfrak{L}$, i.e., $\mathfrak{L}_\alpha^\perp = \mathfrak{L}\backslash\mathfrak{L}_\alpha$ for any $\alpha = \mathrm{cl, SP, UP, CP, qu, DHB, vN}$.

We have

$$\mathfrak{L}_\alpha^{(\perp)} \cap \mathfrak{L}_\beta^{(\perp)} \neq \varnothing \qquad \text{for any } \alpha, \beta = SP, CP, UP, \alpha \neq \beta \tag{10}$$

where the brackets indicate that the sign $\perp$ may or may not occur. Our system of 12 set-theoretic equations (10) reveals the complete logical independence of our notions of SP, CP, and UP. In particular, the equations of the form $\mathfrak{L}_\alpha \cap \mathfrak{L}_\beta^\perp \neq \varnothing$ $(\alpha \neq \beta)$ express the statements that these notions are not logical consequences of each others. Moreover, the equations $\mathfrak{L}_\alpha \cap \mathfrak{L}_\beta \neq \varnothing$ indicate that our formulations of SP, CP, and UP do not contradict each others.

We also have

$$\varnothing \neq \mathfrak{L}_{vN} \subset \mathfrak{L}_{DHB} \subset \mathfrak{L}_\alpha \subset \mathfrak{L}_{qu} \subset \mathfrak{L}_{cl}^\perp, \mathfrak{L}_{cl} \subset \mathfrak{L}_{qu}^\perp \qquad (\alpha = \mathrm{SP, CP, UP}) \tag{11}$$

with all set inclusions being proper. First of all, these relations indicate that our formulations of the three principles SP, CP, and UP are of quantal nature, each of them excluding classical descriptions. This guarantees the consistency of the proposed quantizations. Moreover, in the mutual exclusiveness of the sets $\mathfrak{L}_{\mathrm{cl}}$ and $\mathfrak{L}_{\mathrm{qu}}$ we can read the mutual exclusiveness of the two possibilities: either $h=0$ or $h>0$. Confronting the fact that $\mathfrak{L}_{\mathrm{vN}} \subset \mathfrak{L}_{\mathrm{DHB}}$ with the assumption that the SP, CP, and UP jointly exhaust the quantum principle we realize that *the restriction of* $\mathfrak{L}$ *to* $\mathfrak{L}_{DHB}$ *expresses the proper* (*or full*) *quantization of the abstract* (**L**, **S**) *theory*. This indicates also that the structure of the proper quantum theory is coded in the set $\mathfrak{L}_{\mathrm{DHB}}$, whereas the further restriction of $\mathfrak{L}$ to $\mathfrak{L}_{\mathrm{vN}}$ is just a mathematically convenient restriction of the proper quantum theory $\mathfrak{L}_{\mathrm{DHB}}$.

Finally, we recall that

$$(\mathfrak{L}_{cl} \cup \mathfrak{L}_{qu})^{\perp} \neq \varnothing \tag{12}$$

indicating the abundance of the descriptions (**L**, **S**) resulting from the general considerations. We leave it open whether some prestructures (**L**, **S**) in $(\mathfrak{L}_{\mathrm{cl}} \cup \mathfrak{L}_{\mathrm{qu}})^{\perp}$ may lead to some physically interesting generalizations of the standard classical and quantal theories.

We may now summarize our considerations as follows.

Quantum theory is the theory of quanta. Whenever one wants to have a quantum theory from a given physical theory the universal quantum of action should be incorporated, either explicitly or implicitly, into that theory; i.e., the theory should be quantized. Within the quantum logic approach to axiomatic quantum mechanics a natural way to do that is to introduce the superposition principle, the uncertainty principle, and the complementarity principle into the framework. This leads to a natural characterization of those quantum logics (**L**, **S**) which may provide quantum descriptions of some physical systems. Assuming that the three quantum principles exhaust the fundamental quantum principle, the existence of the universal quantum of action h, we end with singling out those quantum logics which may provide proper quantum descriptions. The given characterization of quantum logics, though not complete, adds also to our understanding of the conceptual foundations of the traditional quantum theory.

REFERENCES

Beltrametti, E. G., and Cassinelli, G. (1976). "Logical and Mathematical Structures of Quantum Mechanics," *Rivista del Nuovo Cimento*, **6**, 321–404.

Beltrametti, E. G., and Cassinelli, G. (1977). "On State Transformations Induced by Yes–No Experiments, in the Context of Quantum Logic," *Journal of Philosophical Logic*, **6**, 369–379.

Beltrametti, E. G., and Cassinelli, G. (1979). "Properties of States in Quantum Logic," *Problems in the Foundations of Physics*, 29–70 (Proceedings of the International School of Physics "Enrico Fermi," Course 72), North-Holland, Amsterdam.

Bohr, N. (1928). "The Quantum Postulate and the Recent Development of Atomic Theory," *Nature*, **121**, 580–590.

Bohr, N. (1935). "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" *Physical Review*, **48**, 696–702.

Bub, J. (1979). "Conditional Probabilities in Non-Boolean Possibility Structure," *The Logico-Algebraic Approach to Quantum Mechanics*, D. Reidel, Dordrecht, Holland, Vol. II, pp. 209–226.

Bugajska, K., and Bugajski, S. (1973). "Description of Physical Systems," *Reports on Mathematical Physics*, **4**, 1–20.

Bugajski, S. (1979). "The Inner Language of Operational Quantum Mechanics," in *Proceedings of the Workshop on Quantum Logic*, "Ettore Majorana" Center for Scientific Culture, Erice-Trapani, Sicily.

Bugajski, S., and Lahti, P. (1980). "Fundamental Principles of Quantum Theory," *International Journal of Theoretical Physics*, **19**, 499–514.

Cassinelli, G., and Beltrametti, E. G. (1975). "Quantum Logic and Ideal Measurements of First Kind," *Fundamenta Scientiae*, Vol. 28, University Louis Pasteur, Strasbourg.

Cassinelli, G., and Beltrametti, E. G. (1975). "Ideal, First-Kind Measurements in a Proposition-State Structure," *Communications in Mathematical Physics*, **40**, 7–13.

Davies, E. (1976). *Quantum Theory of Open Systems*, Academic Press, London.

Davies, E., and Lewis, J. (1970). "An Operational Approach to Quantum Probability," *Communications in Mathematical Physics*, **17**, 239–260.

Dirac, P. (1958). *The Principles of Quantum Mechanics*, Oxford, Clarendon Press (First edition, 1930).

Edwards, C. (1970). "The Operational Approach to Algebraic Quantum Theory I," *Communications in Mathematical Physics*, **16**, 207–230.

Finkelstein, D. (1963). *The Logic of Quantum Physics*, Transactions of the N.Y. Academy of Science, Vol. 25, pp. 621–637.

Finkelstein, D. (1969). *Matter, Space, and Logic*. Boston Studies in the Philosophy of Science, Vol. 5, pp. 199–215.

Gerjuoy, E. (1973). "Is the Principle of Superposition Really Necessary?" in *Contemporary Research in the Foundations and Philosophy of Quantum Theory*, C. A. Hooker, ed., D. Reidel, Dordrecht, Holland, pp. 114–142.

Gleason, A. (1957). "Measures on the Closed Subspaces of a Hilbert Space," *Journal of Mathematics and Mechanics*, **6**, 885–893.

Greechie, R. (1969). "An Orthomodular Poset with a Full Set of States Not Embeddable in Hilbert Space," *Carribean Journal of Science and Mathematics*, **1**, 15–26.

Greechie, R., and Gudder, S. (1973). "Quantum Logics," in *Contemporary Research in the Foundations and Philosophy of Quantum Theory*, C. A. Hooker, ed., D. Reidel, Dordrecht, Holland, pp. 143–173.

Gudder, S. (1970). "Axiomatic Quantum Mechanics and Generalized Probability Theory," in *Probabilistic Methods in Applied Mathematics*, Bharucha-Reid, ed., Academic Press, New York, pp. 53–129.

Gudder, S. (1977). "Four Approaches to Axiomatic Quantum Mechanics," in *The Uncertainty Principle and Foundations of Quantum Mechanics*, W. C. Price and S. S. Chissick, eds., John Wiley & Sons, Ltd., London, pp. 247–276.

Haag, R., and Bannier, U. (1978). "Comments on Mielnik's Generalized (Non Linear) Quantum Mechanics," *Communications in Mathematical Physics*, **60**, 1–6.

Heisenberg, W. (1927). "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik," *Zeitschrift für Physik*, **43**, 172–198.

Jammer, M. (1974). *The Philosophy of Quantum Mechanics*. McGraw-Hill, New York.

Janowitz, M. (1963). "Quantifiers on Quasi-orthomodular Lattices," Ph.D. dissertation, Wayne State University.

Jauch, J. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Lahti, P. (1979a). "Uncertainty and Complementarity in Axiomatic Quantum Mechanics," Dr. dissertation, University of Turku (to appear in *International Journal of Theoretical Physics*).

Lahti, P. (1979b). "On the inter-relations of the three quantal principles," in *Proceedings of the Workshop on Quantum Logic*, "Ettore Majorana" Center for Scientific Culture, Erice-Trapani, Sicily.

Lahti, P. (1980). "Uncertainty Principle and Complementarity in Axiomatic Quantum Mechanics," *Reports on Mathematical Physics*, **17**, 99–110.

Mackey, G. (1963). *Mathematical Foundations of Quantum Mechanics*, W. A. Benjamin, New York.

Maczynski, M. (1967). "A Remark on Mackey's Axiom System for Quantum Mechanics," *Bulletin del Academie Polonaise des Sciences, Serie des Sciences Mathematiques Astronomiques et Physiques*, **15**, 583–587.

Maczynski, M. (1973). "The Orthogonality Postulate in Axiomatic Quantum Mechanics," *International Journal of Theoretical Physics*, **8**, 353–360.

Mielnik, B. (1968). "Quantum Logic and Evolution," *Annales de l'Institut Henri Poincaré*, **9**, 1–5.

Mielnik, B. (1969). "Theory of Filters," *Communications in Mathematical Physics*, **15**, 1–46.

Mielnik, B. (1974). "Generalized Quantum Mechanics," *Communications in Mathematical Physics*, **37**, 221–256.

Piron, C. (1964). "Quantum Axiomatics," *Helvetica Physica Acta*, **37**, 439–468.

Pool, J. (1968). "Baer*-Semigroups and the Logic of Quantum Mechanics," *Communications in Mathematical Physics*, **9**, 118–141.

Pool, J. (1968). "Semimodularity and the Logic of Quantum Mechanics," *Communications in Mathematical Physics*, **9**, 212–228.

Scheibe, E. (1973). *The Logical Analysis of Quantum Mechanics*, Pergamon Press, Oxford.

Taylor, C. (1973). *Physics Reference Books*, Vol. 3, Richard Clay Ltd., Bungay, Suffolk.

Varadarajan, V. (1968). *Geometry of Quantum Theory I*, Van Nostrand Reinhold Co., New York.

von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey (First edition 1932).

Wick, G., Wightman, A., and Wigner, E. (1952). "The Intrinsic Parity of Elementary Particles," *Physical Review*, **88**, 101–105.